COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on ''Critical behavior of the chain-generating function of self-avoiding walks on the Sierpinski gasket family: The Euclidean limit''

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We refute the claims made by Riera and Chalub [Phys. Rev. E **58**, 4001 (1998)] by demonstrating that they have not provided enough data (requisite in their series expansion method) to draw reliable conclusions about criticality of self-avoiding walks on the Sierpinski gasket family of fractals.

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The self-avoiding walk (SAW) on a lattice is a random walk that must not contain self-intersections. The criticality of SAWs has been extensively studied as a challenging problem in statistical physics on the Euclidean lattices and on fractal lattices as well. Accordingly, the question has been posed whether the critical behavior of SAWs on a Euclidean lattice can be retrieved via a limit of an infinite number of fractals whose properties gradually acquire the corresponding Euclidean values. In this Comment we scrutinize the methods used so far to answer the foregoing question.

The most frequently studied infinite family of finitely ramified fractals appears to be the Sierpinski gasket (SG) family. Each member of the SG family is labeled by an integer $b(2\leqslant b\leqslant\infty)$, and when $b\to\infty$ both the fractal d_f and spectral d_s dimension approach the Euclidean value 2. Concerning the study of the criticality of SAWs, these fractal lattices are perfect objects for an application of the renormalization group (RG) method, due to their intrinsic dilation symmetry (the so-called self-similarity) and their finite ramification. The latter property enables one to construct a finite set of the RG transformations and therefrom an exact treatment of the problem. This treatment was first applied by Dhar [1], for $b=2$, and later it was extended [2] up to $b=8$. The obtained results of the corresponding critical exponents, for the finite sequence $2\leqslant b\leqslant 8$, were not sufficient to infer their relation to the relevant Euclidean values. However, these results inspired the finite-size scaling (FSS) approach to the problem [3], which brought about the prediction that the SAW critical exponents on fractals do not necessarily approach their Euclidean values when $b\to\infty$. Indeed, Dhar [3] found that the critical exponent ν, associated with the SAW end-to-end distance, tends to the Euclidean value 3/4, whereas the critical exponent γ, associated with the total number of distinct SAWs, approaches 133/32, being always larger than the Euclidean value $\gamma_E=43/32$.

The intriguing FSS results motivated endeavors to extend the exact RG results beyond $b=8$. However, since this extension appeared to be an arduous task, a new insight was needed. This insight came from a formulation of the Monte Carlo renormalization group (MCRG) method for fractals [4,5], which produced values for ν and γ up to $b=80$. For $2\leqslant b\leqslant 8$ the MCRG findings deviated, from the exact values, at most 0.03%, in the case of ν, and 0.2% in the case of

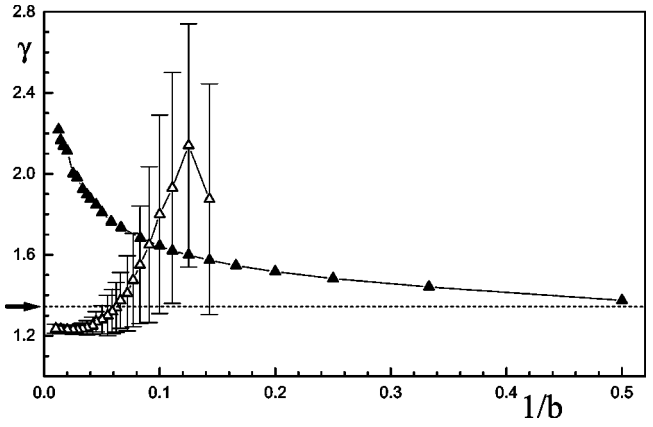

FIG. 1. The SAW critical exponent γ as a function of $1/b$. The solid triangles represent results of the MCRG calculation [5], while the open triangles represent results obtained by Riera and Chalub (RC) via the series expansion method [6]. In both cases the solid lines that connect the data symbols serve as the guide to the eye. One should observe unusually large error bars in the case of RC results, whereas in the case of the MCRG results the error bars lie within the data symbols. The horizontal dashed line represents the Euclidean value $\gamma_E=43/32$ (which is also indicated by the solid horizontal arrow).

TABLE I. The exact RG and MCRG (in the parentheses) results of the PDW critical exponents ν and γ for the SG fractals for $2 \leqslant b \leqslant 100$. The exact RG results have been obtained in [8,9], while the MCRG results are calculated in the present work.

b	ν exact (MC)	γ exact (MC)
2	0.79870(0.79801±0.00075)	0.9520(0.9503±0.0018)
3	0.82625(0.82683±0.00043)	0.9631(0.9638±0.0014)
4	0.84311(0.84332±0.00010)	0.9673(0.9665±0.0011)
5	0.85469(0.85470±0.00008)	0.9695(0.9690±0.0011)
6	0.86329(0.86329±0.00006)	0.9711(0.9707±0.0012)
7	0.87000(0.86992±0.00006)	0.9726(0.9723±0.0012)
8	0.87542(0.87536±0.00005)	0.9742(0.9732±0.0012)
9	0.87992(0.87978±0.00004)	0.9759(0.9749±0.0013)
10	0.88374(0.88370±0.00013)	0.9777(0.9770±0.0014)
12	0.88992(0.88984±0.00011)	0.9815 (0.9812±0.0015)
15	0.89679(0.89662±0.00009)	0.9877 (0.9839±0.0016)
17	0.90035(0.90040±0.00008)	0.9919(0.9911±0.0016)
20	0.90467(0.90460±0.00007)	0.9982(0.9946±0.0018)
25	0.91011(0.91002±0.00006)	1.0084(1.0005±0.0019)
30	0.91417(0.91415±0.00002)	1.0180(1.0181±0.0020)
35	0.91736(0.91727±0.00005)	1.0270(1.0194±0.0022)
40	0.91996(0.91995±0.00005)	1.0354(1.0368±0.0023)
50	0.92399(0.92393±0.00004)	1.0504(1.0457±0.0025)
60	0.92701(0.92704±0.00003)	1.0634(1.0696±0.0026)
70	0.92940(0.92934±0.00003)	1.0750(1.0789±0.0028)
80	0.93136(0.93136±0.00001)	1.0853(1.0854±0.0029)
90	0.93300(0.93300±0.00001)	1.0945(1.0948±0.0031)
100	0.93441(0.93437±0.00002)	1.1029(1.1089±0.0032)

γ. In addition, the behavior of the entire sequence of the MCRG findings, as a function of b, supported the FSS predictions.

Recently, Riera and Chalub [6] made a different type of endeavor to obtain results for the critical exponent γ for large b, by applying an original series expansion method [7]. However, the data of Riera and Chalub (RC) display a quite different behavior than the MCRG results (see Fig. 1). As regards comparison of the RC results with the available exact RG results [2], one may notice a surprising discrepancy: for $b=7$ the RC result deviates 19% (which should be compared with the respective MCRG deviation 0.13%), while for $b=8$ the RC result deviates 33% from the exact result (which is again much larger than the corresponding MCRG deviation 0.15%). On the other hand, concerning the behavior of γ beyond $b=8$, the RC results start to decrease, whereas the MCRG results monotonically increase. Furthermore, Riera and Chalub [6] claimed that, in contrast with the FSS prediction [3], γ should approach the Euclidean value $43/32 = 1.343\,75$, in the limit of very large b. These discrepancies call for inspection of both methods, that is, of the MCRG technique [4,5] and the series expansion method [6,7]. We are going first to reexamine our MCRG approach, and then we shall comment on the applicability of the RC series expansion method for large b.

We have found that the best way to check the validity of the MCRG method, for large b, is to apply it in a case of a random walk model that is exactly solvable for all possible b. To this end, the so-called piecewise directed walk (PDW)

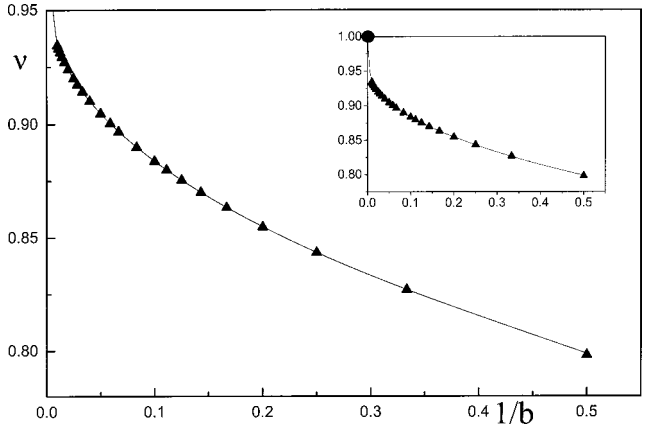

FIG. 2. The PDW critical exponent ν as a function of $1/b$. The solid line represents exact results [8], while the solid triangles represent the MCRG results (see Table 1). The error bars for the MCRG results lie within the data symbols. The inset is given in order to depict the limiting value of ν (solid circle) that coincides with Euclidean value $\nu_E=1$.

[8,9] has turned out to be quite appropriate. The PDW model describes such a random self-avoiding walk on SG fractals in which the walker is allowed to choose randomly, but self-similarly, a limited number of possible step directions [8]. This model corresponds to the directed random walk on Euclidean lattices, in which case $\nu_E=1$ and $\gamma_E=1$. By applying the exact RG approach, the critical exponent ν and γ for the PDW have been obtained [8,9] for each $b(2 \leqslant b < \infty)$. Moreover, it was demonstrated exactly that ν approaches the Euclidean value $\nu_E=1$, while γ tends to the non-Euclidean value $\gamma=2$, when $b \to \infty$. Here, we apply the MCRG method (used in the case of a SAW in [4,5]) to calculate ν and γ of the PDW model for $2 \leqslant b \leqslant 100$. Our results, together with the exact findings, are presented in Table I and depicted in Figs. 2 and 3. One can see that, in the entire region under study, the agreement between the MCRG results and the exact data is excellent. Indeed, the deviation of the MCRG results for ν from the corresponding exact results is at most

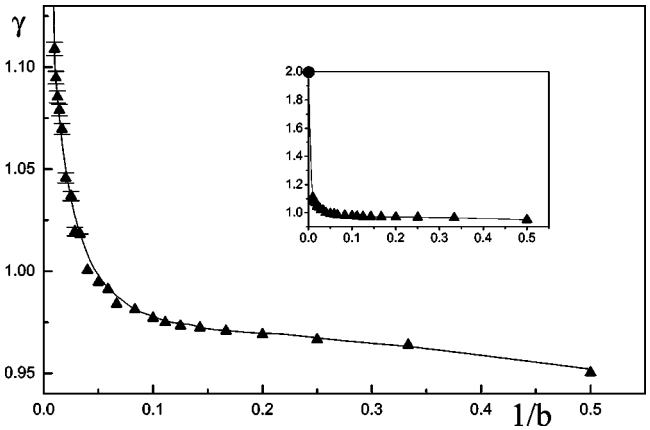

FIG. 3. The PDW critical exponent γ as a function of $1/b$. The solid line represents exact results [9], while the solid triangles represent the MCRG results (see Table I). The data error bars for the MCRG results for b up to 30 are invisible, while for larger b the error bars are comparable to the size of the symbols. The inset is given in order to depict the limiting value $\gamma=2$ (solid circle) which is different from the Euclidean value $\gamma_E=1$.

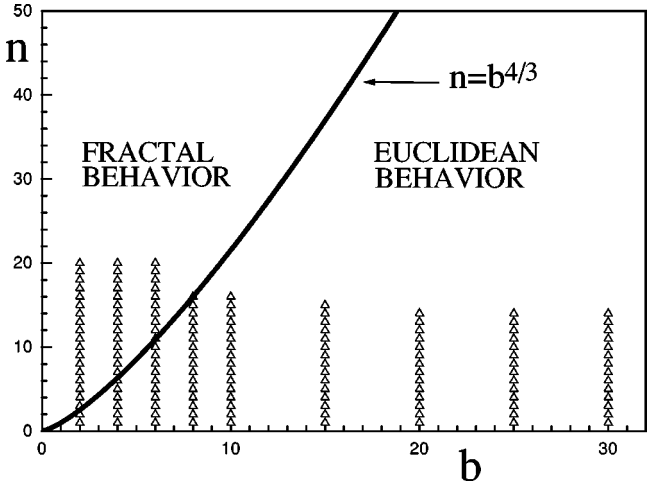

FIG. 4. The schematic representation of the fractal and Euclidean behavior of SAWs on the SG fractals. Regions of the two different behaviors are separated by the solid line $n=b^{4/3}$. The height of each vertical line (comprised of small triangles) corresponds to the maximum length of the n-step SAWs enumerated in the series expansion approach by Riera and Chalub [6] for a given b. In order to study the criticality of SAWs on the SG fractals it is necessary to extend the vertical lines beyond the solid curve. Accordingly, one should observe that results of Riera and Chalub [6] do not probe the fractal region for $b>8$.

0.08%, while in the case of γ it is at most 0.8%. This test of the MCRG method provides novel reliability for its application in studies of random walks on SG with large b.

Because of the confirmed reliability of the MCRG method, and because Riera and Chalub [6] have obtained quite different results, in the case of SAWs on SG, we have reason to assume that their conclusions were obtained in an incorrect way. Thus, we may pose the question, what was wrong in the application of the series expansion method in the work of Riera and Chalub [6]? Let us start with mentioning that in the series expansion study of the SAW the first task is to determine the number $c_n(b)$ of all possible SAWs for a given number n of steps, where $1\leqslant n\leqslant n_{max}$. Of course, in practice, it is desirable to perform this enumeration for very large n_{max}, as the corresponding numbers $c_n(b)$ of all n-step SAWs represent coefficients of the relevant generating function $C_b(x)=\Sigma_{n=1}^{\infty}c_n(b)x^n$ (where x is the weight factor for each step), whose singular behavior determines critical exponents of SAWs. In order to take into account the existence of the SG lacunarity, the average end-to-end distance of the set of n-step SAWs should be larger than the size of the smallest homogeneous part of the SG fractal [3], that is, n_{max} should be larger than $b^{4/3}$. In a case when the number of steps is smaller than $b^{4/3}$ the corresponding SAWs percieve the underlying fractal lattice as a Euclidean substrate. In order to make it more transpicuous, we present in Fig. 4 the curve $n=b^{4/3}$, which divides the (b,n) plane in two regions so that one of them corresponds to the fractal behavior of n-step SAWs, while the other corresponds to the Euclidean behavior. In the same figure, for a given b, we also depict the number of coefficients (empty small triangles) that were obtained by Riera and Chalub [6] for the corresponding SG fractal, in their series expansion approach. One can see that only for $2\leqslant b\leqslant 8$ did Riera and Chalub generate a sufficient number of coefficients $c_n(b)$ so as to probe the fractal-behavior region (in which the condition $n_{max}\geqslant b^{4/3}$ is satisfied). On the other hand, for $b>8$, in all cases studied, the maximum length n_{max} of enumerated SAWs [6] is not larger than 16, and thereby the corresponding generating functions $C_b(x)$ remain in the domain of the Euclidean behavior (see Fig. 4). For instance, in order to study the criticality of SAWs on the SG fractal, for $b=80$, it is a prerequisite to calculate all coefficients $c_n(b)$ in the interval $1\leqslant n\leqslant n_{max}$, where n_{max} must be larger than $80^{4/3}\approx 339$, which is far beyond the $n_{max}=13$ that was reached in [6] for $b=80$. This explains why the RC results for the SG critical exponent γ (see Fig. 1), with increasing b, wrongly become closer to the Euclidean value 1.343 75.

The problems discussed above, that is, the problems with not long enough SAWs, do not appear in the MCRG study of SAWs on the SG fractals, as the RG method in general takes into account SAWs of all length scales. Incidentally, we would like to mention that in [6] it was erroneously quoted that in the MCRG studies [4,5] one Monte Carlo (MC) realization corresponds to simulation of one SAW. In fact, one MC realization implies simulations of all possible walks on the fractal generator, which appears to be the smallest homogeneous part of the SG fractal. For instance, for the $b=80$ SG fractal, in order to calculate the critical exponent γ, in one MC realization we simulated [5] all n-step SAWs with n ranging between 1 and 3240.

Finally, we would like to comment on the analytical argument, given in [6], which was assumed to support the claim $\lim_{b\to\infty}\gamma=\gamma_E$. One can observe that the corresponding argument does not exploit particular properties of the SAWs studied. Thus, if the argument were valid, it could be applied to other types of SAWs on fractals leading to the same conclusion $\lim_{b\to\infty}\gamma=\gamma_E$. However, the case of the PDW discussed in this Comment is a definite counterexample of the foregoing conclusion, as it was rigorously demonstrated [8,9] that in this case $\lim_{b\to\infty}\gamma\neq\gamma_E$.

In conclusion, let us state that in this Comment we have vindicated the expectation that the MCRG technique for studying the SAW critical exponents on fractals is a reliable method and a valuable tool in discussing the query as to whether the critical behavior of SAWs on a Euclidean lattice can be achieved through a limit of an infinite number of fractals whose properties gradually acquire the corresponding Euclidean values. On the other hand, we have demonstrated that Riera and Chalub [6], in an attempt to answer the mentioned query by applying the series expansion method, have not provided a sufficient number of numerical data for a study of criticality of SAWs on the SG fractals with finite scaling parameters b. Therefore, any inference from such a set of data about the large b behavior of the critical exponent γ cannot be tenable.

We would like to acknowledge helpful and inspiring correspondence with D. Dhar concerning the matter discussed in this Comment.

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